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A symmetry invariant integral on κ-deformed spacetime

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Abstract: In this note we present an approach using both constructive and Hopf algebraic methods to contribute to the not yet fully satisfactory definition of an integral on κ-deformed spacetime. The integral presented here is based on the inner product of differential forms and it is shown that this integral is explicitly invariant under the deformed symmetry structure.

Keywords: Non-Commutative Geometry, Quantum Groups.

Contents

1. Introduction

The κ-deformed spacetime is frequently discussed as one of the most attractive models for a noncommutative (NC) spacetime, since it can be endowed with a deformed symmetry structure [1 − 5]. Recently, there has been substantial progress in reinterpreting traditional algebraic-geometric concepts in the NC setting [6 − 8]. However, many problems persist. In this note we present constructive methods combined with some Hopf algebraic concepts to contribute to the not yet fully satisfactory definition of an integral [6, 12] for the κ-deformed spacetime. The integral presented here is based on the inner product of differential forms. Thereby this note continues the recent definition of an n-dimensional calculus of one-forms for an n-dimensional κ-deformed space [8]. It is shown that this integral is explicitly invariant under the deformed symmetry structure. The drawback is that this integral is not cyclic (at least not at first sight), therefore it seems not to be useful for gauge theory.

The work is a consequently new approach in a series of previous attempts of other authors to construct an integral invariant under κ-deformed symmetry transformations. Previous works have focused for example [9] on the deformed Fourier theory, to build a wave-packet using an integration invariant under the deformed action of Lorentz transformations. The same paper proposes also a left invariant and a right invariant integration measure; a similar construction is derived in [10], analysing different ordering procedures and the action of translation generators. Again, the construction in [11] is based on a general analysis of an equation of motion with an infinite number of derivatives. The four approaches of those three papers just mentioned differ fundamentally in derivation and result with the results of this paper at hand. It is of course not claimed here that the best or even most useful definition of an invariant (though not cyclic) integral on κ-deformed space has been constructed. The integral of this paper is actually an inner product; this allows a very rigid Hopf-algebraic treatment, but the conceptual limitations of this approach for a general notion of an integral are obvious. Still, the results of this paper are rather unambiguous (except up to ordering procedures) and to all orders.

This note is organised as follows: in section 2 we recall the most important definitions of [6] and [8]. In section 3 we define the integral as an inner product of differential forms. In section 4 we show that this integral is invariant under the deformed symmetry structure. In appendix A we recall the definition of a cyclic integral from [6] and in B we recall the definition of hermitean symmetry generators discussed as well in [6]. This note should be read as a juxtaposition of two different approaches, the one in the main part of this note and the other in the appendix. We conclude that at present neither of the two approaches is without fail, they will have to be combined in the future in an ingenious way.

2. Abstract κ-euclidean algebra

In this note we use the conventions of [6] and [8]. The κ-deformed spacetime (a Lie algebra space with structure constants $C^{\mu\nu}_{\lambda} = (a^\mu \delta^\nu_\lambda - a^\nu \delta^\mu_\lambda)$) is discussed in its euclidean version for simplicity, the characteristic vector a^μ is aligned with the nth direction, therefore

$$[\hat{x}^n, \hat{x}^j] = ia\hat{x}^j\,, \qquad [\hat{x}^i, \hat{x}^j] = 0\,, \qquad \forall i,j \in \{1,\dots,n-1\}\,. \tag{2.1}$$

There is a deformed Poincaré symmetry structure on this spacetime:

$$\begin{aligned}
[\hat{M}^{rs}, \hat{x}^n] &= 0\,, & [\hat{M}^{rs}, \hat{x}^j] &= \delta^{rj}\hat{x}^s - \delta^{sj}\hat{x}^r\,,\\
[\hat{N}^l, \hat{x}^n] &= \hat{x}^l + ia\hat{N}^l\,, & [\hat{N}^l, \hat{x}^j] &= -\delta^{lj}\hat{x}^n - ia\hat{M}^{lj}\,,\\
[\hat{D}_n, \hat{x}^n] &= \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\,, & [\hat{D}_n, \hat{x}^j] &= ia\hat{D}_j\,,\\
[\hat{D}_i, \hat{x}^n] &= 0\,, & [\hat{D}_i, \hat{x}^j] &= \delta^j_i\Big(-ia\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\Big)\,,
\end{aligned} \tag{2.2}$$

however, this deformed symmetry is undeformed in the algebraic sector. There are two n-dimensional bases of differential one-forms. One of them transforms vector-like under rotations, but has difficult commutation relations with coordinates $(\mu, \nu = 1, \dots, n)$

$$[\hat{\xi}^\mu, \hat{x}^\nu] = ia(\delta^{\mu n}\hat{\xi}^\nu - \delta^{\mu\nu}\hat{\xi}^n) + (\hat{\xi}^\mu\hat{D}_\nu + \hat{\xi}^\nu\hat{D}_\mu - \delta^{\mu\nu}\hat{\xi}^\rho\hat{D}_\rho)\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\lambda\hat{D}_\lambda}\,, \tag{2.3}$$

and one which commutes with coordinates, but transforms under rotations as[1]

$$\begin{aligned}
[\hat{M}^{rs}, \hat{\omega}^n] &= 0\,, \qquad [\hat{M}^{rs}, \hat{\omega}^j] = \delta^{rj}\hat{\omega}^s - \delta^{sj}\hat{\omega}^r\,,\\
[\hat{N}^l, \hat{\omega}^n] &= \hat{\omega}^l\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu} + ia(\hat{\omega}^l\widetilde{\partial}_n - \hat{\omega}^n\widetilde{\partial}_l)\,,\\
[\hat{N}^l, \hat{\omega}^j] &= -\delta^{lj}\hat{\omega}^n\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu} + ia(\hat{\omega}^l\widetilde{\partial}_j - \hat{\omega}^j\widetilde{\partial}_l)\,.
\end{aligned} \tag{2.5}$$

[1]The derivatives $\widetilde{\partial}_j$ used in (2.5) are defined as

$$\begin{aligned}
[\widetilde{\partial}_i, \hat{x}^n] &= ia\widetilde{\partial}_i\,, & [\widetilde{\partial}_n, \hat{x}^n] &= \frac{-ia^3\widetilde{\partial}_s\widetilde{\partial}_s\widetilde{\partial}_n + \sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}{1-a^2\widetilde{\partial}_k\widetilde{\partial}_k}\,,\\
[\widetilde{\partial}_i, \hat{x}^j] &= \delta^j_i\,, & [\widetilde{\partial}_n, \hat{x}^j] &= -ia\widetilde{\partial}_j\frac{-ia\widetilde{\partial}_n + \sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}{1-a^2\widetilde{\partial}_k\widetilde{\partial}_k}\,.
\end{aligned} \tag{2.4}$$

Since the calculus of one-forms is n-dimensional, we have the ordinary deRham differential calculus at our disposition. Especially, there is one n-form, the volume form

$$[\hat{M}^{rs}, \hat{\omega}^1 \dots \hat{\omega}^n] = 0\,, \qquad [\hat{N}^l, \hat{\omega}^1 \dots \hat{\omega}^n] = -ia(n-1)\hat{\omega}^1 \dots \hat{\omega}^n \hat{\partial}_l\,. \tag{2.6}$$

There are other derivatives with useful properties. The simplest derivatives are

$$\begin{aligned} [\hat{\partial}_n, \hat{x}^j] &= 0\,, & [\hat{\partial}_n, \hat{x}^n] &= 1\,, \\ [\hat{\partial}_i, \hat{x}^j] &= \delta_i^j\,, & [\hat{\partial}_i, \hat{x}^n] &= ia\hat{\partial}_i\,. \end{aligned} \tag{2.7}$$

All these symmetry generators and forms etc. cannot only be defined in the abstract algebra, but can be realised on ordinary functions, replacing ordinary pointwise multiplication with the $\star$-product. In the symmetric ordering, the summed-up $\star$-product for κ-deformed space reads

$$\begin{aligned} f(x) \star g(x) = \lim_{\substack{y \to x \\ z \to x}} \exp \Bigg(& x^j \partial_{y^j} \left(e^{-i\hbar a \partial_{z^n}} \frac{-i\hbar a \partial_n}{e^{-i\hbar a \partial_n} - 1} \frac{e^{-i\hbar a \partial_{y^n}} - 1}{-i\hbar a \partial_{y^n}} - 1 \right) + \\ & + x^j \partial_{z^j} \left(\frac{-i\hbar a \partial_n}{e^{-i\hbar a \partial_n} - 1} \frac{e^{-i\hbar a \partial_{z^n}} - 1}{-i\hbar a \partial_{z^n}} - 1 \right) \Bigg) f(y) g(z)\,. \end{aligned} \tag{2.8}$$

All symmetry generators can be represented (on ordinary functions with this $\star$-product), e.g.

$$\begin{aligned} N^{\star l} f(x) &= \left(x^l \partial_n - x^n \partial_l + x^l \partial_\mu \partial_\mu \frac{e^{ia\partial_n} - 1}{2\partial_n} - x^\mu \partial_\mu \partial_l \frac{e^{ia\partial_n} - 1 - ia\partial_n}{ia\partial_n^2} \right) f(x), \\ M^{\star rs} f(x) &= (x^s \partial_r - x^r \partial_s) f(x)\,, \\ D_n^\star f(x) &= \Big(\frac{1}{a} \sin(a\partial_n) - \frac{1}{ia\partial_n \partial_n} \partial_k \partial_k (\cos(a\partial_n) - 1) \Big) f(x)\,, \\ D_j^\star f(x) &= \partial_j \left(\frac{e^{-ia\partial_n} - 1}{-ia\partial_n} \right) f(x)\,. \end{aligned} \tag{2.9}$$

Also the forms can be represented, e.g. $\hat{\omega}^\mu \to \mathrm{d}x^\mu$. For details we refer to [6] and [8].

3. Integration of forms

Integrals for physical actions may be formulated as inner products of forms. In commutative physics, actions are often written in terms of the inner product of two differential r-forms ψ and ϕ, using the Hodge-$*$ operator[2]. In the case of an n-dimensional commutative manifold the Hodge-$*$ is defined on an r-form[3]

$$\phi = \frac{1}{r!} \phi_{\mu_1 \dots \mu_r} \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_r}\,, \tag{3.1}$$

as

$$*\phi = \frac{\sqrt{\det g}}{r!(n-r)!} \phi_{\mu_1 \dots \mu_r} \epsilon^{\mu_1 \dots \mu_r}{}_{\nu_{r+1} \dots \nu_n} \mathrm{d}x^{\nu_{r+1}} \wedge \dots \wedge \mathrm{d}x^{\nu_n}\,. \tag{3.2}$$

[2] Note the different symbols for the $\star$-product and the Hodge-$*$.

[3] Conventions are according to [13].

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Here g is the metric defined on the commutative manifold. Recall the identities $*1 = \sqrt{\det g}\ \mathrm{d}^n x$ and $*^2\phi = (-1)^{r(n-r)}\phi$. The inner product of two r-forms is then the integral over the full spacetime times a measure:

$$(\psi, \phi) = \int \psi \wedge *\phi = \frac{1}{r!}\int \mathrm{d}^n x \sqrt{\det g}\ \ \psi_{\mu_1 \dots \mu_r}\phi^{\mu_1 \dots \mu_r}\,. \tag{3.3}$$

Most physically relevant actions such as the Yang-Mills action and the minimally gauge-coupled fermionic action can be formulated in such a language. Locally, gauge potentials are Lie algebra-valued one-forms $A^0 = iA^0_\mu \mathrm{d}x^\mu$. The field strength $F^0_{\mu\nu}$ are components of a Lie algebra-valued two-form, $F^0 = \mathrm{d}A^0 + A^0 \wedge A^0 = iF^0_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, fulfilling the Bianchi identity $\mathrm{d}F^0 + F^0 \wedge A^0 + A^0 \wedge F^0 = 0$.

To be more specific, the Yang-Mills action is of the form:

$$\begin{aligned}(F^0, F^0) &= \mathrm{Tr}\int (iF^0_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) \wedge *(iF^0_{\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma) \\ &= -\frac{1}{2}\mathrm{Tr}\int \mathrm{d}^n x \sqrt{\det g}\, F^0_{\mu\nu}F^{0\mu\nu}\,. \end{aligned} \tag{3.4}$$

Since the κ-deformed space in our ansatz is considered to be flat, it is sufficient to treat spinor fields as fields of form degree zero. We do not need to consider the Dirac derivative as the sum of two Dirac operators acting on the two spin bundles which make up the exterior bundle. Therefore spinorial actions fit into the framework.

In analogy to this phrasing of commutative actions, we now want to formulate NC field theories in the language of forms. Of course, we need to replace all point-wise products with $\star$-products. From equation (3.4) we see that we also need a suitable set of differential forms which can be combined into a volume form. For example in the Yang-Mills action, one of the two two-forms has to be commuted through the field-components $F^0_{\mu\nu}$ in order to be combined into a volume form. Therefore the frame one-forms $\hat{\omega}^\mu$ which have been defined such that they commute with functions (and in the $\star$-product setting can be identified with ordinary one-forms $\hat{\omega}^\mu \rightarrow= \mathrm{d}x^\mu$) do the job.

This means that the NC Yang-Mills action can be written in the following way, commuting frame one-forms to the left ($\overset{\star}{\wedge}$ is simultaneously a wedge and a $\star$-product):

$$\begin{aligned}(F, F) &= \mathrm{Tr}\int (iF_{\mu\nu}\omega^\mu\omega^\nu) \overset{\star}{\wedge} *(iF_{\rho\sigma}\omega^\rho\omega^\sigma) \\ &= -\frac{1}{2}\mathrm{Tr}\int \omega^{\mu_1}\dots\omega^{\mu_n}\, F_{\mu\nu} \star \left(\sqrt{\det g} F^{\mu\nu}\right). \end{aligned} \tag{3.5}$$

The Hodge-$*$ applied to the field strength tensor on the right (or in general to the second differential form) is proportional to $\sqrt{\det g}$. The authors of [14] have found that $\sqrt{\det g}$ can be identified with the measure μ (introduced in [6], discussed once more in appendix A). The measure μ is the pfaffian of the NC structure, given by ($C^{\mu\nu}_\lambda$ are the Lie algebra structure constants of the NC space)

$$\mu = \det{}^{-1/2}(x^\lambda C^{\mu\nu}_\lambda) = \frac{1}{n! 2^n}\epsilon_{\mu_1\mu_2\dots\mu_{2n}}\left(x^\lambda C^{\mu_1\mu_2}_\lambda\right)\cdots\left(x^\lambda C^{\mu_{2n-1}\mu_{2n}}_\lambda\right). \tag{3.6}$$

Since $x^\lambda C_\lambda^{\mu\nu}$ is zero at the origin and not invertible there, the origin has to be excluded for defining μ. Defining [15] in the abstract algebra a radius $\hat{r}$ in the $(n-1)$-dimensional subspace as $\hat{r} = \sqrt{\sum_{i=1}^{n-1} \hat{x}^i \hat{x}^i}$, the derivations $\hat{r}^j \hat{\partial}_j$ and $\hat{\partial}_n$ have ordinary Leibniz rules (since $\hat{x}^j \hat{f}(\hat{x}) = (e^{-ia\hat{\partial}_n} \hat{f}(\hat{x}))\hat{x}^j$). These derivations are identical to the commutative $r^j \partial_j$ and ∂_n. These commutative derivations can be used to construct a commutative metric

$$g = r^{-2} \sum_{i=1}^{n-1} (\mathrm{d}x^i)^2 + (\mathrm{d}x^n)^2 = (\mathrm{d}\ln r)^2 + \mathrm{d}\Omega_{n-2}^2 + (\mathrm{d}x^n)^2 \,, \tag{3.7}$$

with $\mathrm{d}\Omega_{n-2}^2$ the $(n-2)$-dimensional spherical volume element. Therefore $\sqrt{\det g} = r^{-(n-1)}$ $= \mu$ (more exactly, $\sqrt{\det g} = \mu_2$, cp. appendix A).

The measure $\sqrt{\det g} = \mu$ appears as part of the action of the Hodge-$*$ (ϵ-tensor is as usual fully antisymmetric)

$$*(iF_{\rho\sigma}\omega^\rho \omega^\sigma) = \frac{\mu}{2!(n-2)!} F_{\alpha\beta} \epsilon^{\alpha\beta}{}_{\nu_3 \ldots \nu_n} \omega^{\nu_3} \cdots \omega^{\nu_n} \,. \tag{3.8}$$

If μ should play the role of a measure as in the appendix A, it should multiply the volume element. It can be extracted from the second $\star$-multiplicant because of its properties, $x^j \partial_j \mu = -(n-1)\mu$, $\partial_n \mu = 0$. This leaves additional derivatives ∂_n acting on the two differential forms. We expand up to second order (for two arbitrary r-forms ψ and ϕ):

$$\begin{aligned}
\psi \star (\mu\phi) = {} & \mu\psi\phi + \frac{ia}{2}\mu\big(\partial_n\psi x^j \partial_j \phi - x^j \partial_j \psi \partial_n \phi\big) - \frac{ia}{2}(n-1)\mu \partial_n \psi \phi - \\
& - \frac{a^2}{8}\mu\big(\partial_n^2 \psi x^j x^k \partial_j \partial_k \phi - x^j \partial_j \partial_n \psi x^k \partial_k \partial_n \phi + x^j x^k \partial_j \partial_k \psi \partial_n^2 \phi\big) + \\
& + \frac{a^2}{4}(n-1)\mu\big(\partial_n^2 \psi x^j \partial_j \phi - x^j \partial_j \partial_n \psi \partial_n \phi\big) - \\
& - \frac{a^2}{8}(n-1)n\mu\partial_n^2\psi\phi + \frac{a^2}{12}(n-1)\mu\partial_n^2\psi\phi - \frac{a^2}{12}(n-1)\mu\partial_n\psi\partial_n\phi + \cdots \,.
\end{aligned} \tag{3.9}$$

Under an integral allowing partial integration (Stokes' law), the derivatives ∂_n can be combined into one derivative operator (∂_n commutes with the $\star$-product and μ), which we call K:

$$\int \mathrm{d}^n x \, \psi \star (\mu\phi) = \int \mathrm{d}^n x \mu \;\; \psi \star (K\phi) \,. \tag{3.10}$$

Up to second order we find:

$$\begin{aligned}
K & = 1 + \frac{ia}{2}(n-1)\partial_n - \frac{a^2(n-1)(n-2)}{8}\partial_n^2 - \frac{a^2}{12}(n-1)\partial_n^2 + \cdots \\
& = \left(1 + \frac{ia}{2}\partial_n - \frac{a^2}{12}\partial_n^2 - \cdots\right)^{n-1} = \left(\frac{-ia\partial_n}{e^{-ia\partial_n} - 1}\right)^{n-1} \,.
\end{aligned} \tag{3.11}$$

The reason for having identified an expansion up to second order with an all orders expression will become clear in the next section. Continuing the formulation of the action in terms of forms we will rediscover the derivative operator K from an entirely different argument.

Thus, we have constructed a version of the integral, in which the measure function appears naturally outside of the $\star$-product (using $\omega^1 \dots \omega^n = \mathrm{d}^n x$):

$$\begin{aligned}(F,F) &= \mathrm{Tr} \int (iF_{\mu\nu}\omega^\mu\omega^\nu) \star *(iF_{\rho\sigma}\omega^\rho\omega^\sigma) \\ &= \mathrm{Tr} \int (iF_{\mu\nu}\omega^\mu\omega^\nu) \star \left(\frac{\mu}{2!(n-2)!} F_{\alpha\beta}\epsilon^{\alpha\beta}{}_{\nu_3\dots\nu_n}\omega^{\nu_3}\dots\omega^{\nu_n} \right) \\ &= -\frac{1}{2}\,\mathrm{Tr} \int \omega^1 \cdots \omega^n \mu\, F_{\mu\nu} \star \big(KF^{\mu\nu}\big) = -\frac{1}{2}\,\mathrm{Tr} \int \mathrm{d}^n x \mu\, F_{\mu\nu} \big(KF^{\mu\nu}\big)\,, \end{aligned} \tag{3.12}$$

since μ allows to eliminate one $\star$-product (cp. appendix A). Still, we have to understand better the role of the operator K. This is the content of the next section.

4. Invariance of the integral

The definition of the integral in [6] has been motivated to achieve the trace property, invariance under $\mathrm{SO}_a(n)$ rotations has not been a guiding principle in the construction. In contrast we will now investigate the integral of inner product of forms, and will find that this is $\mathrm{SO}_a(n)$-invariant by definition. Since $\mathrm{SO}_a(n)$ is a Hopf algebra, we have to change the usual notion of invariance used in the context of integrals invariant under symmetry *groups*. Invariance of the integral under the action of an operator $\mathcal{V}$ can be formulated in such a way that $\mathcal{V}$ acts on the integral just as on the trivial one-dimensional representation $\mathbb{C}$, an invariant action transforms like a complex number.

With this notion of invariance, we can construct an action from fields which are modules of $\mathrm{SO}_a(n)$ using the inner product integral. If the field $\hat{\psi}$ transforms under $\hat{M}^{\mu\nu}$, then the dual object, i.e. the linear form mapping $\hat{\psi}$ into complex numbers, has to transform under the antipode $S(\hat{M}^{\mu\nu})$. The condition which the antipode of an arbitrary Hopf algebra has to fulfil is [16]

$$\mathsf{m}(S \otimes 1)\Delta = \eta\epsilon\,, \qquad \text{and} \qquad \mathsf{m}(1 \otimes S)\Delta = \eta\epsilon\,. \tag{4.1}$$

Here m denotes the multiplication of two factors of a tensor product, η is the unit embedding $\mathbb{C}$ into $\mathrm{SO}_a(n)$, Δ the coproduct, and ϵ the counit (cp. [8]). We can therefore prove the invariance of an action integral under $\mathrm{SO}_a(n)$. We have to verify that (we choose the convention that the dual space is the factor on the right hand side of the inner product)

$$(\hat{M}^{\mu\nu}\hat{\psi}, \hat{\phi}) = (\hat{\psi}, S(\hat{M}^{\mu\nu})\hat{\phi})\,. \tag{4.2}$$

Writing the inner product for two r-forms ψ and ϕ explicitly, we obtain the condition that (with the Hodge-dual form on the right in the inner product):

$$\int \big(M^{\star\mu\nu}\psi\big) \star (*\phi) = \int \psi \star (S(M^{\star\mu\nu}) * \phi). \tag{4.3}$$

Note that in (4.3) the differential forms contributing to the volume element $\mathrm{d}^n x$ are still split up among the forms ψ and ϕ. In the following, we want to check that this condition is fulfilled for the inner product. The check can be performed explicitly in the $\star$-product setting, using partial integration.

First we repeat the definition of the antipode on symmetry generators:

$$\begin{aligned}
S(\hat\partial_j) &= -e^{-ia\hat\partial_n}\hat\partial_j\,, & S(\hat\partial_n) &= -\hat\partial_n\,, \qquad S(e^{ia\hat\partial_n}) = e^{-ia\hat\partial_n}\,,\\
S(\hat D_j) &= -e^{ia\hat\partial_n}\hat D_j\,, & S(\hat D_n) &= -\hat D_n + ia\hat D_j\hat D_j e^{ia\hat\partial_n}\,,\\
S(\hat M^{rs}) &= -\hat M^{rs}\,, & S(\hat N^l) &= -\hat N^l e^{-ia\hat\partial_n} - ia\hat M^{lk}\hat\partial_k e^{-ia\hat\partial_n} - ia(n-1)\hat\partial_l e^{-ia\hat\partial_n}\,.
\end{aligned} \tag{4.4}$$

The antipode of the coordinates $\hat x^\mu$ is a priori not defined in our approach, since coordinates here are not regarded as finite translations, i.e. as elements of the κ-deformed Euclidean/Poincaré group, the dual Hopf algebra of $\mathrm{SO}_a(n)$. Therefore no coproduct is defined for the coordinates, but formally the commutation relations of $\hat x^\mu$ with an arbitrary function can be interpreted as a coproduct:[4]

$$\begin{aligned}
\hat x^j\hat f(\hat x) &= (e^{-ia\hat\partial_n}\hat f(\hat x))\hat x^j\,, &&\longrightarrow \hat x^j\otimes 1 - e^{-ia\hat\partial_n}\otimes\hat x^j = 0\,,\\
\hat x^n\hat f(\hat x) &= \hat f(\hat x)\hat x^n + (ia\hat x^k\hat\partial_k\hat f(\hat x))\,, &&\longrightarrow \hat x^n\otimes 1 - 1\otimes\hat x^n - ia\hat x^k\hat\partial_k\otimes 1 = 0\,,\\
S(\hat x^j) &= \hat x^j e^{ia\hat\partial_n}\,,\\
S(\hat x^n) &= \hat x^n - ia\hat\partial_k\hat x^k = \hat x^n - ia\hat x^k\hat\partial_k - ia(n-1)\,.
\end{aligned} \tag{4.5}$$

Regarding hermitean conjugation and antipode, there are two particularly problematic operators (cp. appendix B), $\hat N^l$ and $\hat x^n$. For further clarity we define the $\star$-representations $S(\mathcal V^\star) = S(\mathcal V)^\star$, derived from (4.4):

$$\begin{aligned}
S(\partial_j^\star) &= -\partial_j\frac{e^{-ia\partial_n}-1}{-ia\partial_n}\,, \qquad S(\partial_n) = -\partial_n\,, \qquad S(e^{ia\partial_n}) = e^{-ia\partial_n}\,,\\
S(D_j^\star) &= -\partial_j\frac{e^{ia\partial_n}-1}{ia\partial_n}\,, \qquad S(D_n^\star) = -\frac{1}{a}\sin(a\partial_n) + \frac{\partial_k\partial_k}{ia\partial_n\partial_n}(\cos(a\partial_n)-1)\,,\\
S(M^{\star rs}) &= -x^s\partial_r + x^r\partial_s\,,\\
S(N^{\star l}) &= -x^l\partial_n\frac{e^{-ia\partial_n}+1}{2} + x^n\partial_l\frac{e^{-ia\partial_n}-1}{-ia\partial_n} + x^l\partial_k\partial_k\frac{e^{-ia\partial_n}-1}{-2\partial_n} -\\
&\quad - x^k\partial_k\partial_l\frac{e^{-ia\partial_n}-1+ia\partial_n}{ia\partial_n^2} + (n-1)\partial_l\frac{e^{-ia\partial_n}-1}{\partial_n}\,.
\end{aligned} \tag{4.6}$$

Comparing the $\star$-representations $S(\mathcal V)^\star$ with the result of hermitean conjugation $\mathcal V^\star \to \overline{\mathcal V^\star}$ (integrating $\mathcal V^\star$ under an integral fulfilling Stokes' law, cp. appendix B), we find that they are almost identical (the definition of the antipode does not involve complex conjugation $i\to -i$). Of course, for this partial integration we have to employ the integral definition involving the measure μ and the rescaling $\partial_j \to \tilde\partial_j = \partial_j + \rho_j$ (cp. [6] and appendix B). For example:

$$\begin{aligned}
\int\mu\,(\tilde D_j^\star\tilde\psi)\star\tilde\phi &= \int\mu\left(\tilde\partial_j\frac{e^{-ia\partial_n}-1}{-ia\partial_n}\tilde\psi\right)\star\tilde\phi = \int\mu\tilde\psi\star\Big(-\tilde\partial_j\frac{e^{ia\partial_n}-1}{ia\partial_n}\tilde\phi\Big)\\
&= \int\mu\,\tilde\psi\star\Big(-\tilde D_j^\star e^{ia\partial_n}\tilde\phi\Big) = \int\mu\,\tilde\psi\star\Big(S(\tilde D_j^\star)\tilde\phi\Big)\,.
\end{aligned} \tag{4.7}$$

In this realisation of the antipode in terms of partial integration of rescaled partial derivatives almost all operators can be treated in a satisfactory way. However, as in appendix B,

[4]This leads to the same result as derived in the framework of κ-deformed Euclidean/Poincaré group.

$\tilde{N}^{\star l}$ and $\tilde{x}^{\star n}$ do not fit into the framework. The problematic piece is the factor proportional to $(n-1)$:

$$S(\tilde{N}^l)^\star = \cdots + (n-1)\partial_l \frac{e^{-ia\partial_n}-1}{\partial_n} = (n-1)\left(-ia\partial_l - \frac{a^2}{2}\partial_n\partial_l + \ldots\right). \tag{4.8}$$

Although we obtain a factor proportional to $(n-1)$ from partially integrating $\tilde{N}^{\star l}$ (from the term proportional to $x^j\tilde{\partial}_j$)

$$\tilde{N}^{\star l} \xrightarrow{\text{p.i.}} \cdots - (n-1)\partial_l \frac{e^{-ia\partial_n}-1+ia\partial_n}{ia\partial_n^2} = (n-1)\left(-\frac{ia}{2}\partial_l - \frac{a^2}{6}\partial_l\partial_n + \cdots\right), \tag{4.9}$$

this is not the right term for $S(\tilde{N}^l)^\star$. Changing the definition of μ or the rescaling ρ_j to account for the additional terms does not work, since this would spoil the behaviour of other operators under partial integration.

The only possibility to obtain new terms proportional to $(n-1)$ is to fix the (rescaled) antipode $S(\tilde{N}^l)^\star$, by introducing a derivative operator which acts on the coordinate x^n. We need an asymmetrically acting operator K, which is a power series in the derivatives ∂_n (it must not depend on coordinates x^μ or on ∂_j). We define K such that for all $\mathcal{V} \in \mathrm{SO}_a(n)$ (including coordinates) the following equation is valid (for arbitrary r-forms $\tilde{\psi}$ and $\tilde{\phi}$)

$$\int \mu\,(\tilde{\mathcal{V}}^\star\tilde{\psi}) \star (K\tilde{\phi}) \equiv \int \mu\,\tilde{\psi} \star \left(K\big(S(\tilde{\mathcal{V}}^\star)\tilde{\phi}\big)\right). \tag{4.10}$$

To simplify the subsequent calculation, we eliminate the $\star$-product, afterwards we eliminate the measure and the rescaling $\tilde{\mathcal{V}}^\star \to \mathcal{V}^\star$ by introducing the field redefinition $\tilde{\phi} = \mu^{-\frac{1}{2}}\phi$ (according to the prescription in [6], $\mu^{-\frac{1}{2}}$ commutes with K):

$$\int (\mathcal{V}^\star\psi)(K\phi) \equiv \int \psi\,\big(K\big(S(\mathcal{V}^\star)\phi\big)\big). \tag{4.11}$$

The result of the calculation below does not depend on this field redefinition.

From (4.8) and (4.9) follows that the equation that K has to satisfy reads

$$\begin{aligned}
\left[K, \left(-x^n\partial_l\frac{e^{-ia\partial_n}-1}{-ia\partial_n}\right)\right] &\stackrel{!}{=} (n-1)\partial_l\left(\frac{e^{-ia\partial_n}-1}{\partial_n} + \frac{e^{-ia\partial_n}-1+ia\partial_n}{ia\partial_n^2}\right)K\,,\\
\Leftrightarrow \quad \frac{\partial K}{\partial\partial_n} &\stackrel{!}{=} -(n-1)\frac{-ia\partial_n}{e^{-ia\partial_n}-1}\left(\frac{ia\partial_n e^{-ia\partial_n}+e^{-ia\partial_n}-1}{ia\partial_n^2}\right)K\,,\\
\Leftrightarrow \quad K &= c\left(\frac{-ia\partial_n}{e^{-ia\partial_n}-1}\right)^{n-1}.
\end{aligned} \tag{4.12}$$

The solution is unique up to a complex multiplicative factor c which we fix $c=1$, such that $K = 1 + \mathcal{O}(a)$, i.e. a well-behaved commutative limit.

This operator K is the derivative operator that we have guessed as the result of extracting the measure μ from one of the two factors of the $\star$-product. This means that by constructing an action in terms of differential forms with the Hodge-$*$ we have found an action which is at the same time invariant under all $\mathcal{V} \in \mathrm{SO}_a(n)$

$$\left(\hat{\mathcal{V}}\hat{\psi}, \hat{\phi}\right) = \left(\hat{\psi}, S(\hat{\mathcal{V}})\hat{\phi}\right),$$

since

$$
\begin{aligned}
&\int \left(\tilde{\mathcal{V}}^\star(\tilde\psi_{\mu_1\cdots\mu_r}\omega^{\mu_1}\cdots\omega^{\mu_r})\right)\star *(\tilde\phi_{\nu_1\cdots\nu_r}\omega^{\nu_1}\cdots\omega^{\nu_r}) = \\
&\quad = \int(\tilde\psi_{\mu_1\ldots\mu_r}\omega^{\mu_1}\cdots\omega^{\mu_r})\star\left(S(\tilde{\mathcal{V}}^\star) * (\tilde\phi_{\nu_1\cdots\nu_r}\omega^{\nu_1}\cdots\omega^{\nu_r})\right), \\
&\Leftrightarrow \int\mu\left(\tilde{\mathcal{V}}^\star\left(\tilde\psi_{\mu_1\ldots\mu_r}\frac{1}{r!}\omega^{\mu_1}\ldots\omega^{\mu_r}\right)\right)\star\left(K\tilde\phi_{\nu_1\cdots\nu_r}\frac{\epsilon^{\nu_1\cdots\nu_r}{}_{\mu_{r+1}\cdots\mu_n}}{r!(n-r)!}\omega^{\mu_{r+1}}\cdots\omega^{\mu_n}\right) = \\
&\quad = \int\mu\left(\tilde\psi_{\mu_1\cdots\mu_r}\frac{1}{r!}\omega^{\mu_1}\cdots\omega^{\mu_r}\right)\star\left(KS(\tilde{\mathcal{V}}^\star)\left(\tilde\phi_{\nu_1\cdots\nu_r}\frac{\epsilon^{\nu_1\cdots\nu_r}{}_{\mu_{r+1}\cdots\mu_n}}{r!(n-r)!}\omega^{\mu_{r+1}}\cdots\omega^{\mu_n}\right)\right), \\
&\Leftrightarrow \int\left(\mathcal{V}^\star\left(\psi_{\mu_1\cdots\mu_r}\frac{1}{r!}\omega^{\mu_1}\cdots\omega^{\mu_r}\right)\right)\left(K\left(\phi_{\nu_1\cdots\nu_r}\frac{\epsilon^{\nu_1\cdots\nu_r}{}_{\mu_{r+1}\cdots\mu_n}}{r!(n-r)!}\omega^{\mu_{r+1}}\cdots\omega^{\mu_n}\right)\right) = \\
&\quad = \int\left(\psi_{\mu_1\cdots\mu_r}\frac{1}{r!}\omega^{\mu_1}\ldots\omega^{\mu_r}\right)\left(KS(\mathcal{V}^\star)\left(\phi_{\nu_1\cdots\nu_r}\frac{\epsilon^{\nu_1\ldots\nu_r}{}_{\mu_{r+1}\cdots\mu_n}}{r!(n-r)!}\omega^{\mu_{r+1}}\cdots\omega^{\mu_n}\right)\right). \qquad (4.13)
\end{aligned}
$$

The same is valid for the coordinates in the definition (4.5)

$$\int(x^{\overrightarrow{\star}\mu}\psi)(K\phi) = \int\psi\left(KS(x^{\overrightarrow{\star}\mu})\phi\right). \tag{4.14}$$

The last step in the derivation of an invariant integral is to extract from formulae such as (4.13) the one-forms ω^μ and to combine them into the volume form. We have to be careful in performing this step, since $N^{\star l}$ acts non-trivially on the frame one-forms (2.5). We derive the final result in two steps: first we treat the special case of the inner product of two functions, i.e. two zero-forms. The Hodge dual of a function is proportional to the volume form $\mathrm{d}^n x$. According to (2.6) $\mathrm{d}^n x$ transforms as

$$[\hat N^l, \mathrm{d}^n x] = -ia(n-1)\mathrm{d}^n x\,\hat\partial_l\,. \tag{4.15}$$

On the other hand

$$S(\hat N^l) = -\hat N^l e^{-ia\hat\partial_n} - ia\hat M^{lk}\partial_k e^{-ia\hat\partial_n} - ia(n-1)\hat\partial_l e^{-ia\hat\partial_n}\,. \tag{4.16}$$

Since $[\hat M^{rs}, \mathrm{d}^n x] = 0$ and $[\hat\partial_\mu, \hat\omega^\nu] = 0$, we obtain

$$S(\hat N^l)\mathrm{d}^n x = \mathrm{d}^n x\,(-\hat N^l e^{-ia\hat\partial_n} - ia\hat M^{lk}\hat\partial_k e^{-ia\hat\partial_n})\,. \tag{4.17}$$

The term appearing at the right hand side of (4.17) is (in $\star$-product language)

$$-N^{\star l}e^{-ia\partial_n} - iaM^{\star lk}\partial_k^\star e^{-ia\partial_n} = -\overline{N^{\star l}}\,, \tag{4.18}$$

where the bar denotes complex conjugation. Therefore we can equivalently rewrite (4.13) for the case in which ψ and ϕ are two *complex valued* zero-forms:

$$
\begin{aligned}
\int\mathrm{d}^n x(N^{\star l}\psi)(K\overline{\phi}) &= \int(N^{\star l}\psi)(K(\overline{\phi}\,\mathrm{d}^n x)) = \int\psi\left(KS(N^{\star l})(\phi\,\mathrm{d}^n x)\right) \\
&= -\int\mathrm{d}^n x\,\psi\left(K\overline{N^{\star l}\phi}\right). \qquad (4.19)
\end{aligned}
$$

The same steps can be repeated, if ψ and ϕ are r-forms. We may commute ω^μ with the coefficient functions (we regard the case, in which ω^l is in the first factor, the other case is analogous):

$$\begin{aligned}
&\int \left(N^{\star l}\left(\omega^{\mu_1}\ldots\omega^{\mu_r}\frac{1}{r!}\psi_{\mu_1\ldots\mu_r}\right)\right)\left(K\omega^{\mu_{r+1}}\ldots\omega^{\mu_n}\frac{\epsilon^{\nu_1\ldots\nu_r}{}_{\mu_{r+1}\ldots\mu_n}}{r!(n-r)!}\phi_{\nu_1\ldots\nu_r}\right)\\
&=\int \left(\omega^{\mu_1}\ldots\omega^{\mu_r}\frac{1}{r!}\left(N^{\star l}\psi_{\mu_1\ldots\mu_r}-ia(r-1)\partial^\star_l\psi_{\mu_1\ldots\mu_r}\right)\right)\left(K\omega^{\mu_{r+1}}\ldots\omega^{\mu_n}\frac{\epsilon^{\nu_1\ldots\nu_r}{}_{\mu_{r+1}\ldots\mu_n}}{r!(n-r)!}\phi_{\nu_1\ldots\nu_r}\right)\\
&=\int \mathrm{d}^n x\,\left(N^{\star l}\psi_{\mu_1\ldots\mu_r}\right)\left(K\phi^{\mu_1\ldots\mu_r}\right)-ia(r-1)\int \mathrm{d}^n x\,\left(\partial^\star_l\psi_{\mu_1\ldots\mu_r}\right)\left(K\phi^{\mu_1\ldots\mu_r}\right)\overset{!}{=}\\
&\overset{!}{=}\int \left(\omega^{\mu_1}\ldots\omega^{\mu_r}\frac{1}{r!}\psi_{\mu_1\ldots\mu_r}\right)\left(KS(N^{\star l})\omega^{\mu_{r+1}}\ldots\omega^{\mu_n}\frac{\epsilon^{\nu_1\ldots\nu_r}{}_{\mu_{r+1}\ldots\mu_n}}{r!(n-r)!}\phi_{\nu_1\ldots\nu_r}\right)\\
&=\int \left(\omega^{\mu_1}\ldots\omega^{\mu_r}\frac{1}{r!}\psi_{\mu_1\ldots\mu_r}\right)\times\\
&\quad\times\left(K\omega^{\mu_{r+1}}\ldots\omega^{\mu_n}\frac{\epsilon^{\nu_1\ldots\nu_r}{}_{\mu_{r+1}\ldots\mu_n}}{r!(n-r)!}\left(\overline{(-N^{\star l})}\phi_{\nu_1\ldots\nu_r}+ia\big((n-1)-(n-r)\big)\partial^\star_l e^{-ia\partial_n}\phi_{\nu_1\ldots\nu_r}\right)\right)\\
&=\int \mathrm{d}^n x\,\psi_{\mu_1\ldots\mu_r}\left(K\overline{(-N^{\star l})}\phi^{\mu_1\ldots\mu_r}\right)+ia(r-1)\int \mathrm{d}^n x\,\psi_{\mu_1\ldots\mu_r}\left(K\partial^\star_l e^{-ia\partial_n}\phi^{\mu_1\ldots\mu_r}\right).
\end{aligned}\tag{4.20}$$

Partially integrating the term proportionally to $(r-1)$, the result for complex valued forms is:

$$\int \mathrm{d}^n x\,\left(N^{\star l}\psi_{\mu_1\ldots\mu_r}\right)\left(K\overline{\phi^{\mu_1\ldots\mu_r}}\right)=-\int \mathrm{d}^n x\,\psi_{\mu_1\ldots\mu_r}\left(K\overline{N^{\star l}\phi^{\mu_1\ldots\mu_r}}\right).\tag{4.21}$$

This identity is valid by partial integration and taking into account the action on the volume element and the commutation relation with K. From an abstract definition of inner product we have derived a hermitean representation of $N^{\star l}$. More importantly, (4.21) shows that an action defined in terms of forms is invariant under $N^{\star l}$.

All other operators $M^{\star rs}$ and the derivatives $D^\star_\mu$ and $\partial^\star_\mu$ (no tilde) can be treated analogously. The discussion of these operators is straightforward since they commute with K and with the volume element $\mathrm{d}^n x$ and they be partially integrated without harm (since μ has been eliminated).

The integral just defined is obviously not cyclic, since from the outset we have discussed an asymmetric setting: the $\star$-product is not commutative and therefore it matters whether the Hodge-dual form is in the first or in the second place of the inner product. For the Hopf algebra setting, this however is essential: the order in the inner product *must not* be reversible, since the module space and its second dual space, i.e. the dual of the dual space, are not identical. We recall [8] that the square of the antipode is not the identity:

$$S^2(\hat{N}^l)=\hat{N}^l+ia(n-1)\hat{\partial}_l\neq\hat{N}^l\,.$$

The generator $N^{\star l}$ acts in different ways on a space and its second dual. Therefore it is clear that in formulae such as (4.13) we cannot simply partially integrate once more to obtain the action on the second dual space. The construction of the bidual space has to be redone from scratch. We will not repeat the calculations (via partial integration as above), but they give the following result (ψ and ϕ arbitrary r-forms)

$$\int \left(S(N^{\star l})\psi\right)(K\phi)=\int \psi\left(KS^2(N^{\star l})\phi\right).\tag{4.22}$$

This indeed gives the correct result for the algebraic expression of the square of the antipode. Because of this property, derivative operators such as K generally occur for traces for general Hopf algebras [16]. The integral defined with such an operator is called the *quantum trace*.

We have not been able yet to fully understand the usefulness of the quantum trace. The integral over a field $\int \psi(x)$ can sensibly analysed in this way using $*1 \sim \mu \mathrm{d}^n x$. The product of several fields has to be discussed with great care, as usual in the differential form setting. The operator K can be partially integrated onto the other form:

$$\int \mathrm{d}^n x\, \psi \left(\left(\frac{-ia\partial_n}{e^{-ia\partial_n} - 1} \right)^{n-1} \phi \right) = \int \mathrm{d}^n x \left(\left(\frac{ia\partial_n}{e^{ia\partial_n} - 1} \right)^{n-1} \psi \right) \phi \,. \tag{4.23}$$

The most pressing problem of the quantum trace is that a priori it does not allow to formulate gauge invariant actions from gauge covariant lagrangians, since it is not cyclic (cp. appendix A). Still it is possible to formulate a gauge-covariantised version of the quantum trace, since the derivative ∂_n with an ordinary Leibniz rule can be gauged (cp. the procedure in [7]). Similarly, functions of ∂_n (such as K) can be gauged as well by gauging every single derivative and $\star$-multiplying the gauge-covariantised derivatives. Therefore a gauge-covariantised version of K is possible, however, it is difficult to see how a covariantised K still might provide an $\mathrm{SO}_a(n)$-invariant integral. The only possiblity would be to choose a particular gauge for the gauge potential, i.e. the gauge potential corresponding to ∂_n identical to zero.

The upshot of this discussion is that we have presented a new definition for an integral on the κ-deformed spacetime. It is definitely invariant under the deformed symmetry and has a well-defined geometric interpretation. However, it does not yet have all properties that are desirable for a physical integral. Currently, we still have to choose between formulations of the integral which are either not invariant under symmetry transformations (at least at face value) or not gauge invariant (at least at face value). This is the same conclusion as the authors of [12] have recently drawn w.r.t. the cyclic integral that we discuss for reference in the appendix.

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A. Cyclic integral

An integral may be formulated as a linear map of the coordinate algebra $\mathcal{A}_{\hat{x}}$ into the number field on which it is defined:

$$\int : \mathcal{A}_{\hat{x}} \longrightarrow \mathbb{C} \,, \tag{A.1}$$

$$\int (c_1\hat{\psi} + c_2\hat{\phi}) = c_1 \int \hat{\psi} + c_2 \int \hat{\phi} \,, \qquad \forall \hat{\psi}, \hat{\phi} \in \mathcal{A}_{\hat{x}},\, c_i \in \mathbb{C} \,. \tag{A.2}$$

In addition we demand the trace property in this first section of appendix:

$$\int \hat{\psi}\hat{\phi} = \int \hat{\phi}\hat{\psi}\,. \tag{A.3}$$

The trace property implies that the integral is cyclic $\int \hat{\psi}\hat{\phi}\hat{\chi} = \int \hat{\phi}\hat{\chi}\hat{\psi}$. The integral on the abstract algebra has to be realised in terms of an integral over commutative space, such as the Lebesgue integral. Therefore we need a realisation of the integral in the $\star$-product formalism to perform integration explicitly.

An essential property of the integral is that it allows the use of Stokes' theorem; therefore there are also additional restrictions on the space of allowed functions (their total derivatives must vanish).

Provided that all derivatives, which arise due to an expansion of the $\star$-product, could be eliminated by partial integration at every order, it reduces to point-wise multiplication (A.3). Such a procedure is possible for the Moyal-Weyl $\star$-product, but not for an x-dependent $\star$-product. For example the partial integration in the case of κ-deformed space delivers in first order:

$$\begin{aligned}
&\int \mathrm{d}^n x\, \frac{ia}{2}\big((\partial_n\psi(x))(x^j\partial_j\phi(x)) - (x^j\partial_j\psi(x))(\partial_n\phi(x))\big)\\
&\quad\xrightarrow{\text{part. int.}} \frac{ia}{2}\int \mathrm{d}^n x\, \left(\psi(x)\phi(x) - (n-1)(\partial_n\psi(x))\phi(x)\right).
\end{aligned} \tag{A.4}$$

It has been shown in the framework of deformation quantisation of Poisson manifolds [17] that it is always possible to define a measure function $\mu(x)$ (which is part of the volume element) such that the integral of two functions multiplied with the $\star$-product is cyclic. This has been shown in [18] in a constructive way for quantum spaces. The measure for the κ-deformed spacetime has been discussed first in [6] and then in [15] from the deformation quantisation perspective. For an x-dependent $\star$-product $\theta^{\rho\sigma}(x)$ the measure function $\mu(x)$ has to fulfil the condition:

$$\partial_\rho\left(\mu(x)\theta^{\rho\sigma}(x)\right) = 0\,. \tag{A.5}$$

This statement is an all-orders statement (cp. the discussion in [12]). For κ-deformed spacetime (A.5) entails the following conditions on $\mu(x)$:

$$\partial_\rho\big(\mu(x)a(\delta_n^\rho x^\sigma - \delta_n^\sigma x^\rho)\big) = 0 \Rightarrow \partial_n\mu(x) = 0\,, \qquad x^j\partial_j\mu(x) = -(n-1)\mu(x)\,. \tag{A.6}$$

Examples of measures $\mu(x)$ fulfilling (A.6) are

$$\mu_1(x) = \left(\prod_{i=1}^{n-1} x^i\right)^{-1}, \quad \mu_2(x) = \left(\sum_{i=1}^{n-1}(x^i x^i)\right)^{-\frac{(n-1)}{2}}, \quad \mu_k(x) = \left(\sum_{i=1}^{n-1}(x^i)^k\right)^{-\frac{(n-1)}{k}}, \forall k \in \mathbb{N}\,. \tag{A.7}$$

If $\mu(x)$ is given, the integral over the $\star$-product of two functions has the trace property:

$$\int \mathrm{d}^n x\,\mu(x)\,(\psi(x)\star\phi(x)) = \int \mathrm{d}^n x\,\mu(x)\,(\phi(x)\star\psi(x)) = \int \mathrm{d}^n x\,\mu(x)\,\psi(x)\phi(x)\,. \tag{A.8}$$

The measure $\mu(x)$ allows to eliminate any one of the $\star$-products from the $\star$-product of several functions, because of associativity. This allows to cyclically permute under the integral an arbitrary number of $\star$-multiplied functions

$$\int \mathrm{d}^n x\, \mu(x)\, (\psi_1(x) \star \cdots \star \psi_k(x)) = \int \mathrm{d}^n x\, \mu(x)\, (\psi_k(x) \star \psi_1(x) \star \cdots \star \psi_{k-1}(x)). \qquad \text{(A.9)}$$

Therefore gauge covariant lagrangians lead to gauge invariant actions.

B. Hermitean derivative operators

Conjugation † can be defined on the κ-deformed coordinate algebra and also on its symmetry Hopf algebra $\mathrm{SO}_a(n)$ as a formal involution. We demand:

- Consistency with the algebraic structure $([\mathcal{V},\mathcal{W}] - \mathcal{U})^\dagger = 0$, if $[\mathcal{V},\mathcal{W}] - \mathcal{U} = 0$.
- Complex conjugation for numbers.
- Conjugation is an involution $(\mathcal{V}\mathcal{W})^\dagger = \mathcal{W}^\dagger \mathcal{V}^\dagger$.

An operator is hermitean if $\mathcal{V}^\dagger = \mathcal{V}$. For a straight-forward commutative limit, we demand coordinates to be hermitean and derivatives to be anti-hermitean. We can calculate the conjugation properties of the symmetry generators from their representation in terms of $\hat{x}^\mu$ and $\hat{\partial}_\mu$:

$$\begin{aligned}
(\hat{x}^\mu)^\dagger &= \hat{x}^\mu\,, \qquad \left(\hat{\partial}_n\right)^\dagger = -\hat{\partial}_n\,, \qquad \left(\hat{\partial}_j\right)^\dagger = -\hat{\partial}_j\,,\\
\left(\hat{D}_j\right)^\dagger &= \left(\hat{\partial}_j e^{-ia\hat{\partial}_n}\right)^\dagger = \left(e^{ia\hat{\partial}_n^\dagger}\hat{\partial}_j^\dagger\right) = -\hat{D}_j\,,\\
\left(\hat{D}_n\right)^\dagger &= \left(\frac{1}{a}\sin(a\hat{\partial}_n) + \frac{ia}{2}\hat{\partial}_k\hat{\partial}_k e^{-ia\hat{\partial}_n}\right)^\dagger = -\hat{D}_n\,,\\
\left(\hat{M}^{rs}\right)^\dagger &= -\hat{M}^{rs}\,, \qquad \left(\hat{N}^l\right)^\dagger = -\hat{N}^l\,.
\end{aligned} \qquad \text{(B.1)}$$

Thus, formal conjugation can be defined consistently in the abstract algebra. In addition we need to check the conjugation behaviour of the $\star$-representations. Here derivative operators should be conjugated in a concrete sense, using hermitean conjugation implemented by partial integration under the integral.[5] Thus, we call a derivative operator in the representation $\mathcal{V}^\star$ hermitean if

$$\int \mathrm{d}^n x\, \mu\, \overline{\psi} \star \mathcal{V}^\star \phi = \int \mathrm{d}^n x\, \mu\, \overline{\mathcal{V}^\star \psi} \star \phi\,, \qquad \text{(B.2)}$$

under partial integration. For the two simplest derivative operators $\partial_n^\star$ and $\partial_j^\star$ we obtain that although $\hat{\partial}_\mu^\dagger = -\hat{\partial}_\mu$ (and $\overline{\partial_n^\star} = -\partial_n^\star$)

$$\int \mathrm{d}^n x\, \mu\, \overline{\psi}\, (\partial_i^\star \phi) \stackrel{p.i.}{\to} -\int \mathrm{d}^n x\, \mu\, \overline{\partial_i^\star \psi}\, \phi - \int \mathrm{d}^n x\, \partial_i \mu\, \overline{\frac{e^{ia\partial_n} - 1}{ia\partial_n}\psi}\, \phi\,. \qquad \text{(B.3)}$$

[5] The notion of selfadjointness requires careful definitions of the domain of the operators.

The derivative $\partial_j^\star$ is not anti-hermitean, since it acts on the measure μ. The solution to this problem [6] is a rescaling of the derivative $\partial_j \to \tilde{\partial}_j = \partial_j + \rho_j = \partial_j + \frac{(\partial_j \mu)}{2\mu}$. The rescaling factor ρ_j inherits from μ the properties:

$$x^l \partial_l \rho_j = -\rho_j \quad \text{and} \quad \partial_n \rho_j = 0\,. \tag{B.4}$$

For the choices of μ presented in (A.7), we would obtain:

$$\rho_j(\mu_1) = -\frac{1}{2x^j}, \quad \rho_j(\mu_2) = -\frac{n-1}{2} \frac{x^j}{\sum_{i=1}^{n-1} x^i x^i}\,, \quad \rho_j(\mu_3) = -\frac{n-1}{2} \frac{(x^j)^{k-1}}{\sum_{i=1}^{n-1} (x^i)^k}\,. \tag{B.5}$$

However, it is not necessary to specify a particular form for μ or for ρ_j.

With the rescaled derivative $\tilde{\partial}_j$, anti-hermitean derivative operators can be constructed such as $\tilde{\partial}_j^\star$:

$$\tilde{\partial}_j^\star = (\partial_j + \rho_j) \frac{e^{ia\partial_n} - 1}{ia\partial_n}\,. \tag{B.6}$$

This derivative operator $\tilde{\partial}_j^\star$ is anti-hermitean in the sense of (B.2). Similarly, $D_\mu^\star$ are rescaled to be anti-hermitean in the sense of (B.2):

$$\begin{aligned} D_j^\star &\longrightarrow \tilde{D}_j^\star = (\partial_j + \rho_j) \frac{e^{-ia\partial_n} - 1}{-ia\partial_n}\,, \\ D_n^\star &\longrightarrow \tilde{D}_n^\star = \frac{1}{ia\partial_n^2} (\partial_k + \rho_k)(\partial_k + \rho_k)(\cos(a\partial_n) - 1) + \frac{1}{a}\sin(a\partial_n)\,. \end{aligned} \tag{B.7}$$

The rescaling with ρ_j is algebraically consistent [6]: $[(\partial_j + \rho_j), x^\mu] = \delta_j^\mu$ is unchanged and also $[(\partial_i + \rho_i), (\partial_j + \rho_j)] = 0$. Thus, the rescaling can be lifted into the abstract algebra. Then however, the representation of all operators $M^{\star\mu\nu} \to \tilde{M}^{\star\mu\nu}$ and $x^\star \to \tilde{x}^{\star\mu}$ has to be changed as well:

$$\begin{aligned} \tilde{N}^{\star l} &= x^l \partial_n \frac{e^{ia\partial_n} - 1}{2} + x^l \tilde{\partial}_j \tilde{\partial}_j \frac{e^{ia\partial_n} - 1}{2\partial_n} - x^n \tilde{\partial}_l \frac{e^{ia\partial_n} - 1}{ia\partial_n} - x^j \tilde{\partial}_j \tilde{\partial}_l \frac{e^{ia\partial_n} - 1 - ia\partial_n}{ia\partial_n^2}\,, \\ \tilde{M}^{\star rs} &= x^s \tilde{\partial}_r - x^r \tilde{\partial}_s\,, \quad \tilde{x}^{\star n} = \tilde{x}^{\overrightarrow{\star}\, n} = x^n - x^k \frac{\tilde{\partial}_k}{\partial_n} \left(\frac{ia\partial_n}{e^{ia\partial_n} - 1} - 1 \right), \quad \tilde{x}^{\star j} = \tilde{x}^{\overrightarrow{\star}\, j} = x^{\star j}. \end{aligned}$$

The notation $\tilde{x}^{\overrightarrow{\star}\,\mu}$ denotes a coordinate multiplied from the left to another function.

Unfortunately, even including the action of $\tilde{N}^{\star l}$ on $\mathrm{d}^n x$, $\tilde{N}^{\star l}$ is not anti-hermitean in the sense of (B.2). The problematic piece is the one proportional to $x^j \tilde{\partial}_j \tilde{\partial}_l$; this arises due to the representation of $\tilde{x}^{\star n}$ which is not defined as a hermitean quantity in this approach. We have made several attempts to cure this problem, including the definition of the opposite algebra, acting from the right, or an interpretation of hermitean conjugation for coordinates as changing from left to right multiplication. But in none of these attempts, the hermitean conjugation of $\tilde{N}^{\star l}$ is fully satisfactory. With the cyclic integral of appendix A, we cannot find a fully consistent definition of hermitean operators. However, we have shown in the main part of this note, that with the quantum trace this problem can be handled.

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